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Stress boundary conditions for plate bending

Frederic Y.M. Wan

Department of Mathematics, University of California – Irvine, Irvine, CA 92697-3875, USA

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Abstract

The determination of the appropriate boundary conditions for a two-dimensional theory of elastic flat plates (and shells) consistent with the expected order of accuracy of the theory is both critical and challenging. The reciprocal theorem of elasticity will be applied in a novel way to obtain the appropriate stress boundary conditions for plate bending accurate to all order (with respect to the usual dimensionless thickness parameter) for plates of general edge geometry and loading. Kirchhoff's two contracted stress boundary conditions are shown to be consistent with a leading term (thin plate) approximation theory, but the more general results obtained herein are needed for higher order theories.

Keywords: Stress; Boundary conditions; Plate bending

1. Introduction

Even with today's computing power, an accurate numerical solution for three-dimensional elastostatics of plate and shell structures is often impractical and sometime infeasible. When it is practical, a relatively simple approximate analytical solution for the same problem is often desirable as it allows us to see more clearly how the behavior of the structure depends on the relevant design parameters. Plate and shell theories have been developed over the years to provide ways to obtain such approximate solutions away from the edges of thin structures. From the results of Gregory and Wan (1984, 1985a) and Lin and Wan (1988), we have now explicit examples showing that the higher order accuracy offered by the governing differential equations of a higher order plate or shell theory (with respect to a small dimensionless thickness parameter ε) may not be attained unless commensurate boundary conditions are developed and used for these equations. These boundary conditions have now been developed for plate and shell problems with special edge geometries or restricted loading conditions (see Gregory and Wan, 1984, 1985a,b, 1988, 1993; Gregory et al., 1998). The present paper obtains the appropriate boundary conditions for the Levy (interior) solution for plate bending, and hence a two-dimensional plate theory of any order of accuracy, subject to general admissible edge tractions at an edge with a general contour. (The same general technique applies also to other types of admissible edge conditions as well.) The classical Kirchhoff contracted stress boundary

E-mail address: fwan@math.uci.edu (F.Y.M. Wan).

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conditions for thin plates (Love, 1944; Koiter, 1964) are shown to be consistent with the leading term asymptotic approximation of the stress boundary conditions obtained. Modifications of the Kirchhoff contracted conditions by the new and more general results herein will generally be necessary for a higher order plate theory.

2. Interior and boundary layer solution

For the purpose of developing appropriate boundary conditions for the plate bending problem, it suffices to consider a homogeneous, isotropic, linearly elastic plate bounded by two flat faces at $z = \pm h$, and an edge, E, spanning the cylindrical surface $\{f(x, y) = 0,\ -h \leqslant z \leqslant h\}$ (see Fig. 1). The plate is subject to no interior body loading and no surface tractions at the two faces so that

$$\sigma_{zx}(x, y, \pm h) = \sigma_{zy}(x, y, \pm h) = \sigma_{zz}(x, y, \pm h) = 0 \tag{1}$$

for all (x, y) inside the simple edge curve Γ defined by $f(x, y) = 0$, which does not cross itself. Here n and t are in the direction normal and tangent to the edge curve Γ, respectively. The edge E itself is subject to a prescribed set of admissible tractions so that

$$\sigma_{nn} = \bar{\sigma}_{nn}(\theta, z), \quad \sigma_{nt} = \bar{\sigma}_{nt}(\theta, z), \quad \sigma_{nz} = \bar{\sigma}_{nz}(\theta, z), \tag{2}$$

along E, where the barred quantities are the prescribed tractions along the edge E and where θ is an edgewise variable along Γ (such as the angular variable of the polar coordinate system in the case of a circular edge). There are no restriction on the prescribed admissible tractions along E except that they give rise only to plate bending (with the in-plane stress components being odd in z and the transverse shear component being even in z). The case of plate stretching has already been analyzed by Gregory and Wan (1988).

It has been known since the work of Levy (1877) that there is an exact solution of the equations of three-dimensional elasticity theory that is traction free at the two faces of the plate. The expressions for the stress and displacement fields of the plate bending portion of this solution given in terms of a two-dimensional biharmonic (mid-plane transverse displacement) function $w(x, y)$ can be found in Gregory and Wan (1985a). We will give here in polar coordinates only the six quantities needed in the subsequent analysis:

$$u_r^{\mathrm{I}} = -\frac{z}{1-v}\frac{\partial}{\partial r}\left[(1-v) + \left(h^2 - \frac{2-v}{6}z^2\right)\nabla^2\right]w \tag{3}$$

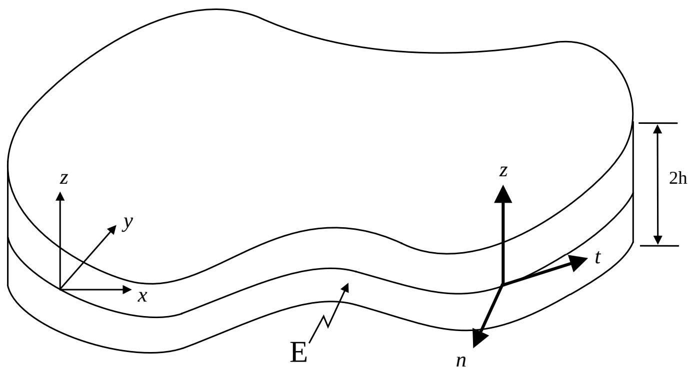

Fig. 1. A simply-connected flat plate.

$$u_\theta^{\mathrm{I}} = -\frac{z}{1-v}\frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-v) + \left(h^2 - \frac{2-v}{6}z^2\right)\nabla^2\right]w \tag{4}$$

$$u_z^{\mathrm{I}} = \left[1 + \frac{vz^2}{2(1-v)}\nabla^2\right]w \tag{5}$$

$$\sigma_{rr}^{\mathrm{I}} = -\frac{Ez}{1-v^2}\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left\{v - \left(h^2 - \frac{2-v}{6}z^2\right)\nabla^2\right\}\right]w \tag{6}$$

$$\sigma_{r\theta}^{\mathrm{I}} = -\frac{Ez}{1-v^2}\left[\frac{\partial}{\partial r} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)(1-v) + \left(h^2 - \frac{2-v}{6}z^2\right)\nabla^2\right]w \tag{7}$$

$$\sigma_{rz}^{\mathrm{I}} = -\frac{E}{2(1-v^2)}(h^2 - z^2)\frac{\partial}{\partial r}\nabla^2 w \tag{8}$$

Here E is Young's modulus of elasticity (distinguished by the context from the same symbol used for the plate edge), v is Poisson's ratio, and

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} \quad \text{with } \nabla^2\nabla^2 w = 0. \tag{9}$$

The general single-valued solution for $w(r,\theta)$ bounded throughout the plate's mid-plane is given by the following Fourier series in the polar angle θ:

$$w(r,\theta) = \sum_{n=1}^{\infty}\{(a_n r^n + b_n r^{n+2})\cos(n\theta) + (c_n r^n + d_n r^{n+2})\sin(n\theta)\} + \{a_0 + b_0 r^2\} \tag{10}$$

Evidently, the Levy solution reduces to the Kirchhoff thin plate (approximate) solution if terms involving $\nabla^2 w$ are omitted in the displacement fields and the in-plane stress fields. (These terms are of higher order in h/a compared to other terms in the same expressions, with a being a representative length of the plate span such as the radius of the circular contour in the case of a circular plate edge.) Similar to the Kirchhoff solution, the Levy solution's simple dependence on the thickness coordinate z makes it generally incapable of satisfying all the prescribed edge conditions along E; hence, it is only a particular solution of the equations of elasticity. The asymptotic analysis of Friedrichs and Dressler (1961), Gol'denveizer (1962), and Gol'denveizer and Kolos (1965) showed that (1) the complete *outer* (*asymptotic expansion of the exact*) *solution* for the equations of three-dimensional elasticity that is stress free on the two plate faces telescopes to become Levy's exact solution, and (2) the residual between the exact solution and the Levy solution consists of only boundary layer phenomena. Such boundary layer residual solution components decay exponentially away from the edge of the plate and become insignificant at a distance large compared to the thickness of the plate. We define an elastostatic state of the plate to be a *boundary layer* (*or a decaying*) *state* if its displacement and stress fields $\{\mathbf{u},\sigma\}$ satisfy the condition

$$\{\mathbf{u},\sigma\} = \mathrm{O}(M\mathrm{e}^{-\gamma a/h}) \quad \text{as } h \to 0 \tag{11}$$

for some maximum modulus M of the prescribed edge tractions, and a positive constant γ. (Though the results of this paper does not depend on the actual value of γ, we know from the various special cases that γ is greater than unity.) An elastostatic state of the plate is said to be a *regular state* if its displacement and stress fields have at worst an algebraic growth (in the parameter $\varepsilon = h/a$) as $h \to 0$. More recently, Gregory (1992) proved that the exact solution of the plate problem is in fact the sum of the Levy solution and boundary layer residual solution components of the Papkovich–Fadle type:

$$\{\mathbf{u}, \sigma\} = \{\mathbf{u}^{\mathrm{I}}, \sigma^{\mathrm{I}}\} + \{\mathbf{u}^{\mathrm{d}}, \sigma^{\mathrm{d}}\} \tag{12}$$

with the superscript 'I' indicating the fact that the elastostatic fields are significant throughout the *interior* of the plate while the superscript 'd' indicating decaying elastostatic states. If truncated asymptotic expansions in a small thickness parameter $\varepsilon = h/a$ should be used for the various elastic fields, we would indicate this by

$$\{\mathbf{u}, \sigma\} \sim \{\mathbf{u}^{\mathrm{o}}, \sigma^{\mathrm{o}}\} + \{\mathbf{u}^{\mathrm{L}}, \sigma^{\mathrm{L}}\} \tag{13}$$

where the superscripts 'o' (for *outer*) and 'L' (for *layer*) indicate truncated asymptotic expansions of the exact solution with respect to the (previously defined) parameter ε. It is important to note here that $\{\mathbf{u}^{\mathrm{L}}, \sigma^{\mathrm{L}}\}$ is generally not smaller than $\mathrm{O}(\varepsilon)$ compared to $\{\mathbf{u}^{\mathrm{o}}, \sigma^{\mathrm{o}}\}$.

The combination of interior and decaying solution components is known to be capable of fitting any admissible set of prescribed edge conditions along the plate edge E (Gregory, 1979, 1980a,b, 1992). Hence boundary value problems of the elastostatics of flat plates are solved in principle. In fact, examples of such solutions for specific boundary value problems can be found in Gregory and Wan (1984), Lin and Wan (1993) and Gregory et al. (1997, 1998, 2001). However, for most plate geometries and edge loadings, solutions by eigenfunction expansions of the decaying component are also often not feasible or practical. We are therefore forced to return to the Kirchhoff type approach of determining only the exact or asymptotic interior solution *without* calculating the boundary layer solution components as well. In this context, the following question arises naturally: What portion of the prescribed edge data should be assigned to the interior solution (with the balance going to the residual decaying solution components)? Equivalently, what conditions must the residual edge data satisfy in order for it to induce only a decaying solution state? The answer to either one of these two questions would enable us to determine the interior solution of the plate without any reference to the supplementary boundary layer states which are much more difficult to obtain.

For the case of stress data prescribed along the edge E, such an assignment of edge data is usually made by means of a modified form of Saint Venant's principle requiring that the Levy type plate theory solution and the prescribed edge stresses have the same transverse shear resultant and (the bending and twisting) moment resultants (Love, 1944; Timoshenko and Goodier, 1951; Reissner, 1963). We put aside for the moment the issue that the three resultant conditions are one too many for the interior solution governed by the fourth order biharmonic equation and the appropriateness of the resolution of this predicament by Kirchhoff's contraction of the three stress boundary conditions into two conditions. There are still a number of questions and issues regarding this modified Saint Venant principle approach that need to be addressed:

First, Saint Venant's principle is, strictly speaking, not applicable to our plate bending problem since the typical linear span of the loaded area is not small compared to the characteristic dimension of the plate. More specifically, the circumference of the loaded edge E is not small compared to a representative span of the plate. Why then should a modified form of this principle, taking resultants across thickness only, be expected to be appropriate for our plate problem?

Second, Saint Venant's principle itself was proved only for some very restricted geometries and loading conditions which do not include the plate bending problem (see Horgan and Knowles, 1983; Horgan, 1989). If Saint Venant's conjecture or its modified form is expected to be applicable to the plate problem, can we prove that it is in fact "appropriate" and deduce also the kind of accuracy (in terms of the small parameter h/a) we can expect from this approach?

Third, if the edge conditions are specified in terms of the displacement components or a mixture of stresses and displacements, Saint Venant's principle, even if it should be applicable, is not useful (since not all edge stresses are known); how do we make the proper assignment of the edge data in these cases?

Finally, we recall that the Kirchhoff contracted stress boundary conditions were originally obtained by a physical argument. Later, Kelvin and Tait derived them mathematically by the direct method of calculus of variations on the basis of certain assumed displacement distributions across the plate thickness (Thomson and Tait, 1867; Love, 1944). More recently, Koiter show that the (exact) two-dimensional differential equations of equilibrium (for both plates and shells) may be re-written, without any approximation, in terms of a set of alternative stress resultant and couple variables such four of these correspond to the four prescribed stress measures of the four Kirchhoff contracted boundary conditions. It is clear from Koiter's analysis that, even from a strictly mathematical viewpoint, the solution of Kirchhoff's plate theory provides only an approximation of a "(more) complete" two-dimensional plate theory (such as a plate theory with transverse shear deformability, see Reissner, 1985). In relation to the exact three-dimensional elasticity solution, do these contracted conditions in fact lead to a degree of accuracy in the approximate solution consistent with the accuracy expected of the Kirchhoff plate theory or, more generally, of the particular set of plate equations adopted for the analysis?

Some of these questions have been answered for special classes of problems in references cited above. They will be addressed in the following sections for the general plate bending problem considered herein.

3. Reciprocal theorem of elasticity

A general approach has been developed by Gregory and Wan (1984, 1985a, 1988) to properly assign a portion of the edge data to the interior solution to avoid a simultaneous determination of the boundary layer solution components for thin elastic bodies. The approach is based on the reciprocal theorem of elasticity. We take the first state in this theorem to be the exact solution of the plate bending problem and denote it by a superscript '(1)'. For the second state, denoted by superscript '(2)', we take a solution of the elasticity equations for the same plate domain (possibly leaving out a small neighborhood of a solution singularity) with no body loads, no face tractions at $z = \pm h$, and *no edge tractions along E*. With these two elastostatic states, we apply the reciprocal theorem to a portion of our plate from the edge E inward to the edge of the a fictitious circular hole of radius ρ centered at a point (possibly a state 2 singularity) sufficiently far away from the edge E, with a minimum distance $d \gg h$ (see Fig. 2). In that case, the reciprocal relation takes the form

$$\int\int_E \{\bar{\sigma}_{nn}u_n^{(2)} + \bar{\sigma}_{nt}u_t^{(2)} + \bar{\sigma}_{nz}u_z^{(2)}\}\,\mathrm{d}S = \int\int_{E_\rho} \{[\sigma_{rr}u_r^{(2)} + \sigma_{r\theta}u_\theta^{(2)} + \sigma_{rz}u_z^{(2)}] - [\sigma_{rr}^{(2)}u_r + \sigma_{r\theta}^{(2)}u_\theta + \sigma_{rz}^{(2)}u_z]\}\,\mathrm{d}S \tag{14}$$

Note that (i) there are no volume integrals in this relation because there are no body loads in the plate interior; (ii) there are no surface integrals over the faces because of the traction free conditions of both

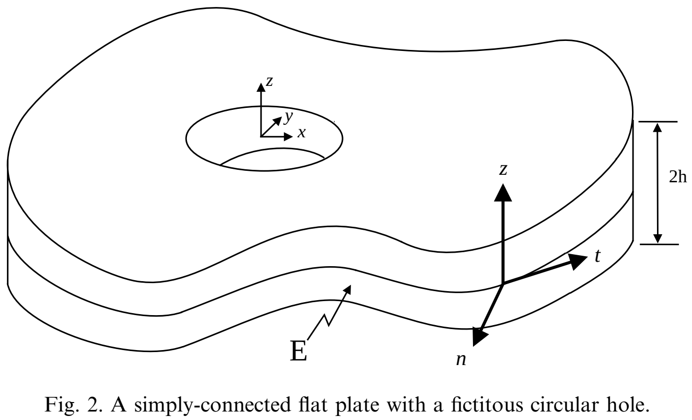

Fig. 2. A simply-connected flat plate with a fictitous circular hole.

elastostatic states on these faces; (iii) a second surface integral over E on the left vanishes because of the traction free conditions of state (2) along E; (iv) the superscript (1) for state (1) has been omitted on the right side to emphasize the fact state (1) is the exact solution of the original problem; and (v) the stress components of state (1) along E on the left must be equal to the prescribed edge stresses and have been written in terms of these prescribed quantities instead.

Except for three rigid body displacements and rotations, the only possible non-trivial state (2) for no interior loading that is also stress free on the two faces and along the edge E must be singular somewhere in the interior of the plate. There are four families of such singular solutions with their only singularity at the center of the fictitious circular hole, $r = 0$, generated by the following singular biharmonic mid-plane transverse displacement fields by way of Levy's formulas for stresses and displacements:

$$w^{kc}(r,\theta) = W_0^{kc} r^{-k} \cos(k\theta), \quad w^{ks}(r,\theta) = W_0^{ks} r^{-k} \sin(k\theta) \tag{15}$$

$$w^{k'c}(r,\theta) = W_0^{k'c} r^{-k+2} \cos(k\theta), \quad w^{k's}(r,\theta) = W_0^{k's} r^{-k+2} \sin(k\theta) \tag{16}$$

where the multiplicative constants W_0^{mq} may be chosen as appropriate factors that make the four families of generating function dimensionless. For a circular plate of radius a, we may choose $W_0^{kc} = W_0^{ks} = a^k$ and $W_0^{k'c} = W_0^{k's} = a^{k-2}$. However, since our final results do not depend on the dimension of the singular biharmonic functions, we will set $W_0^{kc} = W_0^{ks} = W_0^{k'c} = W_0^{k's} = 1$ in all subsequent developments for simplicity.

For a fixed $k \geqslant 2$, any one of these four singular biharmonic (mid-plane transverse displacement) functions induces a singular interior solution of the Levy type. The relevant Levy's stress components by themselves do not satisfy the traction free conditions required of state (2) along E. The singular Levy solution will have to be supplemented by additional Levy solutions of the type induced by the general bounded transverse displacement $w(r,\theta)$ in the form of Eq. (10) as well as decaying solution components to meet the free edge requirements. We denote the resulting state (2) by $\{\mathbf{\Sigma}^S, \mathbf{U}^S\}$ and the singular portion of these (2) states by $\{\sigma^*, u^*\}$ with $* = kc, k'c, ks$ or $k's$ where $k' = k-2$, depending on our choice of the generating *singular* mid-plane transverse displacement. In that case, the reciprocal relation (14) may be written as

$$\int\int_E \{\bar{\sigma}_{nn} U_n^S + \bar{\sigma}_{nt} U_t^S + \bar{\sigma}_{nz} U_z^S\}\,\mathrm{d}S = \int\int_{E_\rho} \{[\sigma_{rr} U_r^S + \sigma_{r\theta} U_\theta^S + \sigma_{rz} U_z^S] - [u_r \Sigma_{rr}^S + u_\theta \Sigma_{r\theta}^S + u_z \Sigma_{r\theta}^S]\}\,\mathrm{d}S \tag{17}$$

This form of the reciprocal relation also applies to $k = 1$ though two families of the "singular" solutions in (15) corresponding k' are no longer singular for this case. They correspond to rigid body rotations and give rise to no stresses throughout the plate, qualifying them for (2) states without any adjustment by supplementary decaying states and bounded interior states. The special case of $k = 0$ will be discussed in the next section.

One other requirement will have to be imposed on state (2) for our method of solution. All (2) states should be a *regular* elastostatic state as defined in the previous section so that its displacement and stress fields have at worst an algebraic growth as $h \to 0$. We now summarize the specification of state (2) as follows:

- It satisfies the homogeneous governing equations of three-dimensional theory of elasticity in the same plate domain as state (1) (taken to be the solution of our original problem).
- It is stress free on the two faces.
- It is stress free along the edge E.
- It has at most one singularity at a minimum distance d away from the edge E large compared to the plate thickness.
- It is a regular elastostatic state with at worst an algebraic growth as $h \to 0$.

With the circular hole $r = \rho$ far away from the edge E, i.e., $h \ll d$, the decaying solution components of both state (1) and state (2) are exponentially small for $r \leqslant \rho$. Both states being also regular states, each is equal to its interior solution component (including the singular component) except for exponentially small terms (EST). Furthermore, for a very small hole, the singular portion dominates the interior component of the (2) state. Altogether, the reciprocal relation (17) simplifies to

$$\int\int_E \{\bar{\sigma}_{nn} U_n^S + \bar{\sigma}_{nt} U_t^S + \bar{\sigma}_{nz} U_z^S\} \, dS$$
$$= \int_{-h}^{h} \int_0^{2\pi} \{[(\sigma_{rr}^{\mathrm{I}} U_r^* + \sigma_{r\theta}^{\mathrm{I}} U_\theta^* + \sigma_{rz}^{\mathrm{I}} U_z^*) - (u_r^{\mathrm{I}} \Sigma_{rr}^* + u_\theta^{\mathrm{I}} \Sigma_{r\theta}^* + u_z^{\mathrm{I}} \Sigma_{rz}^*)]_{r=\rho}\} \rho \, d\theta \, dz \tag{18}$$

Thus, for any of the singular (2) state corresponding to $* = kc, k'c, ks$ or $k's$ with $k = 1, 2, 3, \ldots$, the Levy interior solution is related to prescribed stress data along E by the relation (18) except for exponentially small terms. (As indicated earlier, the $k = 0$ case will be discussed in the next section.) It might appear that by replacing $(\mathbf{\Sigma}^S, \mathbf{U}^S)$ by $(\mathbf{\Sigma}^*, \mathbf{U}^*)$, there would be an additional error of the order of $(\rho/d)^m$ for some m; but this is not the case as we will show later that the relation (18) is independent of ρ to the singular (2) states of interest.

4. Determination of the interior solution

We now make use of the reciprocal relation (18) to determine the Fourier coefficients $\{a_k, b_k, c_k, d_k\}$ in the expansion for $w(r, \theta)$ in (10), and hence Levy's interior solution for our problem. Let $(\mathbf{\Sigma}^*, \mathbf{U}^*)$ be the singular interior state generated by $w^{kc}(r, \theta) = r^{-k} \cos(k\theta)$ and $(\sigma^{\mathrm{I}}, \mathbf{u}^{\mathrm{I}})$ be the Levy interior solution component of the exact solution of our original problem generated by the general (non-singular) biharmonic function (10). Orthogonality among the trigonometric functions eliminates all terms on the right-hand side of (18) except those with a multiplicative factor $\cos(k\theta)$. Since the dependence on z and θ are now explicit for both $(\mathbf{\Sigma}^*, \mathbf{U}^*)$ and $(\sigma^{\mathrm{I}}, \mathbf{u}^{\mathrm{I}})$, we can carry out the integration with respect to both z and θ on the right-hand side of (18) giving us one relation between a_k and b_k. A second relation for a_k and b_k can be obtained by using the other singular solution $w^{k'c}(r, \theta) = r^{-k+2} \cos(k\theta)$, i.e., with $* = k'c$. Together they determine a_k and b_k since the $(\mathbf{\Sigma}^S, \mathbf{U}^S)$ states associated with w^{kc} and $w^{k'c}$ needed on the left-hand side of (18) are known.

With the help symbolic manipulation software (e.g., Maple), the rather involved calculations on the right-hand side of (18) were carried out quickly and error free. It turns out that for $* = kc$, corresponding to the first generating function in (15), the reciprocal relation (18), which is exact up to EST, is independent of ρ, the radius of the fictitious circular hole. Furthermore, it is a relation for b_k alone:

$$\int\int_E \{\bar{\sigma}_{nn} U_n^S + \bar{\sigma}_{nt} U_t^S + \bar{\sigma}_{nz} U_z^S\}_{*=kc} \, dS = -8\pi D k(k+1) b_k \tag{19}$$

where $D = 2Eh^3/3(l - \nu^2)$ is the *bending stiffness factor* of the plate (with the factor 2/3 changed to the more familiar 1/12 if plate thickness is changed from $2h$ to h).

The second relation obtained with $* = k'c$, corresponding to the first generating function in (16), is also independent of ρ but now involved both a_k and b_k:

$$\int\int_E \{\bar{\sigma}_{nn} U_n^S + \bar{\sigma}_{nt} U_t^S + \bar{\sigma}_{nz} U_z^S\}_{*=k'c} \, dS = 8\pi D k(k-1) \left[a_k + b_k \frac{4(4+\nu)(k+1)h^2}{5(1-\nu)} \right] \tag{20}$$

The contribution of the b_k term in the relation above is small of order $(h/d)^2$ and may be neglected for a moderately thick plate theory (with an error of the same order).

The fact that the right-hand side of the reciprocal relation (18) is independent of ρ for both $* = kc$ and $k'c$ implies that the relations (18)–(20) all hold for all ρ $(\leqslant d)$, in particular for $\rho = 0$. For this choice of ρ, the bounded portion of the Levy solution in $(\mathbf{\Sigma}^S, \mathbf{U}^S)$ is absent and the replacement of the singular state $(\mathbf{\Sigma}^S, \mathbf{U}^S)$ in the relation (17) by $(\mathbf{\Sigma}^*, \mathbf{U}^*)$ maintains the same accuracy (which is exact up to EST) as that when only the decaying state components are omitted.

The formulas (19) and (20) and the corresponding formulas for c_k and d_k (obtained by setting $* = ks$ and $k's$) completely determine the Levy solution for the original stress boundary value problem without simultaneously determining the boundary layer solution components. It should be clear from the solution process that the same method for finding the Fourier coefficients of the biharmonic generating function w (10) of the relevant Levy solution applies to other set of admissible edge conditions as well. The only modification consists of changing the edge conditions for state (2) at E to the homogeneous counterpart of the actual prescribed edge conditions. If all edge conditions of the actual problem are given in terms of the displacement components, then all displacement components of state (2) should vanish along E in order for the left-hand side of the relevant reciprocal relation corresponding to (14) to be in terms of the prescribed displacement data and the known singular state (2). In this paper, we will focus only on the case where all the edge conditions are prescribed in terms of the stress components.

Except for cases involving a rigid body displacement field, the construction of a singular (2) state, $(\mathbf{\Sigma}^S, \mathbf{U}^S)$, generally requires the use of the boundary layer solution components in order for it to satisfy the stress free conditions along E. However, such a singular state (2) is a canonical problem to be solved only once and the result applies to all stress boundary value problems for the same plate whatever the actual prescribed edge stress distributions may be. It can be generated numerically if necessary as long as the numerical solution is accurate to the same order of accuracy as the plate theory, i.e., the truncated outer asymptotic solution, used. As we shall see in the next two sections, it is possible, for a few classes of stress boundary value problems, to circumvent the the inclusion of the boundary layer solution components in $(\mathbf{\Sigma}^S, \mathbf{U}^S)$ either completely (as in Section 5) or for a (Kirchhoff type) thin plate approximation (see Section 6).

5. Axisymmetric deformation of a circular plate

For solutions single-valued in θ, the two singular biharmonic functions for mid-plane transverse displacement for the $k = 0$ case are

$$w^0(r) = \ln(r), \quad w^{0'}(r) = r^2 \ln(r) \tag{21}$$

while the corresponding generating function for the general non-singular axisymmetric Levy interior solution $w(r)$ is as given by the $n = 0$ portion of Eq. (10):

$$w_0 = a_0 + b_0 r^2 \tag{22}$$

Note that the terms associated with the Fourier coefficient a_0 corresponds to a rigid body vertical translation and therefore gives rise to no stresses throughout the plate. A unit vertical translation may be used as a (2) state in the reciprocal theorem as it satisfies all the conditions specifying a (2) state in Section 3. For such a (2) state, the reciprocal relation (18) for the case of a *circular edge of radius a* simplifies to

$$\int_0^{2\pi} \int_{-h}^{h} \{\bar{\sigma}_{rs} \cdot 1\} a \, \mathrm{d}z \, \mathrm{d}\theta = \int_{-h}^{h} \int_0^{2\pi} \{\sigma_{rz}^{\mathrm{I}} \cdot 1\}_{r=\rho} \rho \, \mathrm{d}\theta \, \mathrm{d}z \tag{23}$$

or, since $\sigma_{rz}^{\mathrm{I}} = 0$ for the Levy solution generated by Eq. (22),

$$2\pi a \int_{-h}^{h} \bar{\sigma}_{rz} \, \mathrm{d}z \equiv 2\pi a \overline{Q}_r = 0 \tag{24}$$

The condition (24) is just the requirement that the prescribed transverse shear stress distribution along the plate edge must have no resultant axial force so that the plate is in overall equilibrium. Otherwise, there is no other applied load to balance the axial force.

Parenthetically, we note that a non-self-equilibrating distribution of $\bar{\sigma}_{rz}$ that generates a resultant axial force must be balanced by an equal and opposite axial force in the interior of the plate (such as that due to a distribution of axisymmetric transverse edge shear at an inner edge or a point force at the center of the circular plate). For the case of a concentrated load at the origin, the Levy solution for w_0 now includes a singular term $b_{0'}\ln(r)$. The rigid body displacement (2) state in (23) now determines the Fourier coefficient $b_{0'}$:

$$b_{0'} = \frac{a\overline{Q}_r}{4D} \tag{25}$$

Returning to the original problem with a bounded interior solution generated by (22), another (2) state that determines b_0 by way of (18) is obtained with the first generating function $w^0(r) = \ln(r)$ in (21). For this singular generating function, the relevant stress and displacement components of the Levy solution are

$$\Sigma^S_{rr} = E\frac{z}{a}\left[1 - \frac{a^2}{r^2}\right], \quad \Sigma^S_{r\theta} = \Sigma^S_{rz} = 0 \tag{26}$$

$$U^S_r = z\left[(1-v)\frac{r}{a} + (1+v)\frac{a}{r}\right], \quad U^S_\theta = 0 \tag{27}$$

$$U^S_r = -\frac{1}{2}\frac{r^2}{a}(1-v) - (1+v)a\ln\left(\frac{a}{r}\right) - v\frac{z^2}{r} \tag{28}$$

With this (2) state, the reciprocal relation (18) for a *circular edge of radius a* becomes

$$-2Db_0 = \int_{-h}^{h}\left[z\bar{\sigma}_{rr} - \frac{vz^2}{2a}\bar{\sigma}_{rz}\right]\mathrm{d}z \tag{29}$$

As with the $k > 0$ cases, we have now two exact $(\boldsymbol{\Sigma}^S, \mathbf{U}^S)$ states for the axisymmetric bending problem. Both of them do not involve any boundary layer solution components. However, they determine only the Fourier coefficient b_0 but not the rigid body transverse displacement component a_0. Such a level of non-uniqueness is expected when all edge data are prescribed in terms of stresses. In addition to determining b_0, we also recovered the overall equilibrium requirement.

With the following sole non-trivial stress component along the edge $r = a$ generated by the axisymmetric mid-plane transverse displacement (22),

$$\sigma^{\mathrm{I}}_{rr} = -\frac{2Ez}{1-v}b_0 \tag{30}$$

the left-hand side of (29) is just the axisymmetric bending *moment resultant* (also referred to as the bending *stress couple*) M_{rr} (conventionally defined as $z\sigma^{\mathrm{I}}_{rr}$ integrated across the plate thickness) so that the relation (29) may be written as a boundary condition on the bending moment resultant at the plate edge:

$$M_{rr}(r = a) = \int_{-h}^{h}\left[z\bar{\sigma}_{rr} - \frac{vz^2}{2a}\bar{\sigma}_{rz}\right]\mathrm{d}z \tag{31}$$

An equivalent form of the boundary condition (31), expressed as a necessary condition for the residual boundary data to induce a boundary layer solution state, was previously obtained in Gregory and Wan (1985a).

The two stress boundary conditions (24) and (31) for plate bending resulting from our use of the reciprocal theorem are accurate up to exponentially small terms (because we have replaced $\{\sigma_{rr}, \sigma_{rz}\}$ by $\{\sigma^{\rm I}_{rr}, \sigma^{\rm I}_{rz}\}$). It is significant then that the condition (31) on the plate bending moment resultant is not identical to the conventional condition upon the application of (a modified form of) the Saint Venant principle (Love, 1944; Timoshenko and Goodier, 1951; Reissner, 1963). (Note that condition (31) coincides with that obtained by Reiss (1962) if we take $\bar{\sigma}_{rz} = 0$ as Reiss did in his paper. However, $\bar{\sigma}_{rz}$ needs not vanish identically even if it should be self-equilibrating.) At the same time, deviation from the conventional condition consists only of the new term $-(\nu z^2/2a)\bar{\sigma}_{rz}$; its contribution to the boundary condition is generally small of order h/a compared to the first term. In other words, the conventional stress boundary conditions for plate bending is consistent with the Kirchhoff thin plate approximation. But if we wish to retain the additional accuracy of any higher order plate theory, the condition (31) should be used.

Counter-examples have also been constructed by Gregory and Wan (1985a) to illustrate the appropriateness of condition (31) for higher order plate theories. It was shown there that certain three-dimensional decaying elastostatic states for the axisymmetric plate bending problem are consistent with (31) but not with the conventional bending moment condition (corresponding to (31) without the second term). It was also shown that the residual solution induced by the residual data after an assignment by (31) for the Levy solution is a decaying state while a similar solution induced by the residual data after an assignment by the conventional bending moment condition is not.

6. Circular plate with unsymmetric edge stresses

For plate bending without axisymmetry, it is generally not possible to avoid the inclusion of decaying solution components in the determination of the four relevant (2) states. An important exception is the case of stress edge conditions in the Kirchhoff thin plate approximation of the Levy interior solution. To elucidate, it suffices to consider a circular plate with edge stresses along the circular boundary depending on the edge variable θ in the form of $\cos(k\theta)$ or $\sin(k\theta)$ for $k \geqslant 2$ so that they are self-equilibrating. The plate response to the edge loads will also have the same dependence on θ. For the (2) state generated by the first singular mid-plane displacement of (15), the following three stress components are required to vanish at $r = a$:

$$\frac{\sigma^{kc}_{rr}}{\cos(k\theta)} = -\frac{Ez}{1-\nu^2}\left\{ b^s_k\left[(k+1)\langle(k+2)-\nu(k-2)\rangle r^k + 4k(k^2-1)r^{k-2}\left(h^2 - \frac{2-\nu}{6}z^2\right)\right] + a^s_k k(k-1)(1-\nu)r^{k-2} + k(k+1)(1-\nu)r^{-k-2}\right\} + \hat{\sigma}^{kc}_{rr}(r,z) \tag{32}$$

$$\frac{\sigma^{kc}_{r\theta}}{\sin(k\theta)} = \frac{Ez}{1+\nu}\left\{ b^s_k k(k+1)\left[r^k + \frac{4(k-1)}{1-\nu}r^{k-2}\left(h^2 - \frac{2-\nu}{6}z^2\right)\right] + a^s_k k(k-1)r^{k-2} - k(k+1)r^{-k-2}\right\} + \hat{\sigma}^{kc}_{r\theta}(r,z) \tag{33}$$

$$\frac{\sigma^{kc}_{rz}}{\cos(k\theta)} = -\frac{2E}{1-\nu^2}(h^2 - z^2)\{b^s_k k(k+1)r^{k-1}\} + \hat{\sigma}^{kc}_{rz}(r,z) \tag{34}$$

where $\hat{\sigma}^{kc}_{ij}(r,z)$ is the (r,z)-dependent portion of the boundary layer components of σ^{kc}_{rz}. The corresponding displacement components will be needed for the evaluation of the boundary integrals (19) and (20) for a_k and b_k. For our purpose, we will write these expressions in terms of the relevant mid-plane transverse displacement w^{kc}:

$$w^{kc} = \{a_k^s r^k + b_k^s r^{k+2} + r^{-k}\}\cos(k\theta) \tag{35}$$

with

$$\begin{aligned} u_z^{kc} &= \left[1 + \frac{\nu z^2}{2(1+\nu)}\nabla^2\right] w^{kc} + \hat{u}_z^{kc}\cos(k\theta) \\ &= \left\{ b_k^s \left\langle r^{k+2} + \frac{2\nu z^2}{1-\nu}(k+1)r^k \right\rangle + a_k^s r^k + r^{-k} + \hat{u}_z^{kc}(r,z) \right\}\cos(k\theta) \end{aligned} \tag{36}$$

$$\begin{aligned} u_r^{kc} &= -\frac{z}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w^{kc} + \hat{u}_r^{kc}\cos(k\theta) \\ &= \left\{ -z\left[b_k^s \left\langle (k+2)r^{k+1} + \frac{4k(k+1)}{1-\nu} r^{k-1}\left(h^2 - \frac{2-\nu}{6}z^2\right)\right\rangle + a_k^s k r^{k-1} - k r^{-k-1}\right] + \hat{u}_r^{kc}(r,z)\right\}\cos(k\theta) \end{aligned} \tag{37}$$

$$\begin{aligned} u_\theta^{kc} &= -\frac{z}{1-\nu}\frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu) + \left(h^2 - \frac{2-\nu}{6}z^2\right)\nabla^2\right] w^{kc} + \hat{u}_\theta^{kc}\sin(k\theta) \\ &= \left\{ z\left[b_k^s \left\langle k r^{k+1} + \frac{4k(k+1)}{1-\nu} r^{k-1}\left(h^2 - \frac{2-\nu}{6}z^2\right)\right\rangle + a_k^s k r^{k-1} + k r^{-k-1}\right] + \hat{u}_\theta^{kc}(r,z)\right\}\sin(k\theta) \end{aligned} \tag{38}$$

We know from the asymptotic analyses of Friedrichs and Dressler (1961), Gol'denveizer (1962), Reiss (1962), Gol'denveizer and Kolos (1965) and others that the decaying components of the above state (2) displacement fields, $\{\hat{u}_z^{kc}\cos(k\theta), \hat{u}_r^{kc}\cos(k\theta), \hat{u}_\theta^{kc}\sin(k\theta)\}$, are at least $O(h/a)$ smaller in magnitude than the leading term of their interior counterpart $\{w^{kc}, -z\partial w^{kc}/\partial r, -(z/r)\partial w^{kc}/\partial\theta\}$. Roughly, because state (2) is stress free along E with $\sigma_{ry}^{kc}(a,\theta,z) = 0$, for $y = r, \theta$, and z, the decaying components $\{\hat{\sigma}_{rr}^{kc}, \hat{\sigma}_{r\theta}^{kc}, \hat{\sigma}_{rz}^{kc}\}$ in (32)–(34) are of the same order of magnitude as their interior counterpart. Now stresses are obtained by differentiating the displacement fields and differentiation does not change the order of magnitude of the interior solution components but increases the magnitude of the decaying components by a factor proportional to a/h. Therefore, we have except for terms of the order of h/a,

$$u_z^{kc} \sim w^{kc} = \{a_k^s r^k + b_k^s r^{k+2} + r^{-k}\}\cos(k\theta) \tag{39}$$

$$u_r^{kc} \sim -z\frac{\partial}{\partial r}w^{kc} = -z\{a_k^s k r^{k-1} + b_k^s(k+2)r^{k+1} - k r^{-k-1}\}\cos(k\theta) \tag{40}$$

$$u_\theta^{kc} \sim -\frac{z}{r}\frac{\partial}{\partial r}w^{kc} = kz\{a_{k'}^s r^{k-1} + b_{k'}^s r^{k+1} + r^{-k-1}\}\sin(k\theta) \tag{41}$$

Upon substituting these asymptotic expressions into (19), we obtained the following leading term approximation for the Fourier coefficient b_k of the Levy solution of our original stress boundary value problem:

$$\begin{aligned} -8\pi D k(k+1) b_k &= \int_0^{2\pi}\int_{-h}^{h} [\bar{\sigma}_{rr}U_r^{kc} + \bar{\sigma}_{r\theta}U_\theta^{kc} + \bar{\sigma}_{rz}U_z^{kc}]_{r=a}\, a\,\mathrm{d}z\,\mathrm{d}\theta \\ &\sim \int_0^{2\pi}\left[-\overline{M}_{rr}\frac{\partial w^{kc}}{\partial r} - \overline{M}_{r\theta}\frac{1}{r}\frac{\partial w^{kc}}{\partial\theta} + \overline{Q}_r w^{kc}\right]_{r=a} a\,\mathrm{d}\theta \end{aligned} \tag{42}$$

Keeping in mind that the prescribed stresses are proportional to $\cos(k\theta)$ or $\sin(k\theta)$ for $k \geqslant 2$,

$$\{\bar{\sigma}_{rr}(\theta,z), \bar{\sigma}_{rz}(\theta,z)\} = \{\bar{\sigma}_{rr}^k(z), \bar{\sigma}_{rz}^k(z)\}\cos(k\theta), \quad \bar{\sigma}_{r\theta}(\theta,z) = \bar{\sigma}_{r\theta}^k(z)\sin(\theta) \tag{43}$$

we can integrate the right-hand side of the expression for b_k to get

$$b_k \sim \frac{a}{8Dk(k+1)}\left[\overline{M}^k_{rr}\frac{\mathrm{d}\hat{w}^{kc}}{\mathrm{d}r} - \overline{V}^k_r \hat{w}^{kc}\right]_{r=a} \tag{44}$$

where

$$w^{kc}(r,\theta) = \hat{w}^{kc}(r)\cos(k\theta), \quad V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} \tag{45}$$

$$\{\overline{M}^k_{rr}, \overline{M}^k_{r\theta}\} = \int_{-h}^{h}\{\bar{\sigma}^k_{rr}(z), \bar{\sigma}^k_{r\theta}(z)\}z\,\mathrm{d}z, \quad \overline{Q}^k_r = \int_{-h}^{h}\bar{\sigma}^k_{rz}(z)\,\mathrm{d}z \tag{46}$$

Similarly, we have from the first generating function in (16)

$$u_z^{k'c} \sim w^{k'c} = \{a^s_{k'}r^k + b^s_{k'}r^{k+2} + r^{-k+2}\}\cos(k\theta) \tag{47}$$

$$u_r^{k'c} \sim -z\frac{\partial}{\partial r}w^{k'c} = -z\{a^s_{k'}kr^{k-1} + b^s_{k'}(k+2)r^{k+1} - (k-2)r^{-k-1}\}\cos(k\theta) \tag{48}$$

$$u_\theta^{k'c} \sim -\frac{z}{r}\frac{\partial}{\partial\theta}w^{k'c} = kz\{a^s_{k'}r^{k-1} + b^s_{k'}r^{k+1} + r^{-k-1}\}\sin(k\theta) \tag{49}$$

To a leading term asymptotic approximation, we get from (20)

$$a_k \sim \frac{a}{8Dk(k-1)}\left[\overline{V}^k_r\hat{w}^{k'c} - \overline{M}^k_{rr}\frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r}\right]_{r=a} \tag{50}$$

where we have omitted the b_k term on the left-hand side of (20) because it is $\mathrm{O}(h^2/a^2)$ compared to the dominant terms in the same equation. With a_k and b_k given by (44) and (50), we have now the interior stresses and displacements in terms of the prescribed edge loads at least to a leading term approximation (equivalent to a Kirchhoff type theory). This was accomplished without doing any calculations that involve the decaying components of the exact solution.

7. Kirchhoff's contracted stress boundary conditions

The leading term of the Levy solution for our plate problem is now seen from the expression for the Fourier coefficients a_k and b_k, (44) and (50), to depend only on the *bending moment resultant* $\overline{M}_{rr}$ and the *effective transverse shear resultant* $\overline{V}_r$ (see (45)) just as the solution of Kirchhoff 's thin plate theory. At the same time, the contraction of the prescribed stresses of the problem to two resultant quantities as proposed by Kirchhoff occurs naturally in the solution process. However, we cannot yet conclude that the stress boundary conditions for a leading term Levy solution are in fact the same as those in the conventional thin plate theory.

Suppose we compute from the Levy solution the corresponding moment resultants and transverse shear resultant which appear in conventional two-dimensional plate theories. For a *leading term asymptotic approximation*, we have

$$M_{rr} = \int_{-h}^{h}\sigma^{\mathrm{I}}_{rr}z\,\mathrm{d}z \sim -D\{a_k k(k-1)(1-\nu)r^{k-2} + b_k(k+1)[(k+2) - \nu(k-2)]r^k\}\cos(k\theta) \tag{51}$$

$$M_{r\theta} = \int_{-h}^{h} \sigma_{r\theta}^{\mathrm{I}} z\,\mathrm{d}z \sim D(1-\nu)\{a_k k(k-1)r^{k-2} + b_k k(k+1)r^k\}\sin(k\theta) \tag{52}$$

$$Q_r = \int_{-h}^{h} \sigma_{rz}^{\mathrm{I}}\,\mathrm{d}z \sim -4Db_k k(k+1)r^{k-1}\cos(k\theta) \tag{53}$$

We now use (44) and (50) to express the Fourier coefficients a_k and b_k of the interior solution of our original problem in terms of the prescribed boundary data to get *to a leading term approximation*

$$\begin{aligned}\frac{M_{rr}(a,\theta)}{a^{k-1}\cos(k\theta)/8k} = {} & \overline{M}_{rr}^{k}\left[k(1-\nu)\frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r} - \{(k+2)-\nu(k-2)\}a^2\frac{\mathrm{d}\hat{w}^{kc}}{\mathrm{d}r}\right]_{r=a} \\ & - \overline{V}_{r}^{k}[k(1-\nu)\hat{w}^{k'c} - \{(k+2)-\nu(k-2)\}a^2\hat{w}^{kc}]_{r=a}\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{V_r(a,\theta)}{a^{k-2}\cos(k\theta)/8} = {} & \overline{V}_{r}^{k}[k(1-\nu)\hat{w}^{k'c} + (4-k+k\nu)a^2\hat{w}^{kc}]_{r=a} \\ & - \overline{M}_{rr}^{k}\left[k(1-\nu)\frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r} + (4-k+k\nu)a^2\frac{\mathrm{d}\hat{w}^{kc}}{\mathrm{d}r}\right]_{r=a}\end{aligned} \tag{55}$$

In general, the value of the bending moment resultant and the transverse shear resultant at the edge $r = a$ of the circular plate would depend on both $\overline{M}_{rr}^{k}$ and $\overline{V}_{r}^{k}$. In order to have these relations identical to the two Kirchhoff contracted stress boundary conditions, we need to have

$$[k(1-\nu)\hat{w}^{k'c} - \{(k+2)-\nu(k-2)\}a^2\hat{w}^{kc}]_{r=a} = 0 \tag{56}$$

$$\left[k(1-\nu)\frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r} + (4-k+k\nu)a^2\frac{\mathrm{d}\hat{w}^{kc}}{\mathrm{d}r}\right]_{r=a} = 0 \tag{57}$$

$$\left[k(1-\nu)\frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r}\{(k+2)-\nu(k-2)\}a^2\frac{\mathrm{d}\hat{w}^{kc}}{\mathrm{d}r}\right]_{r=a} = 8ka^{1-k} \tag{58}$$

$$[k(1-\nu)\hat{w}^{k'c} + (4-k+k\nu)a^2\hat{w}^{kc}]_{r=a} = 8a^{2-k} \tag{59}$$

with $\hat{w}^{kc}$ and $\hat{w}^{k'c}$ given in (35) and (47), respectively. To evaluate the left-hand side of these four relations, we need to know the coefficients $\{a_k^s, b_k^s, a_{k'}^s, b_{k'}^s\}$ in $\hat{w}^{kc}$ and $\hat{w}^{k'c}$ (see (35) and (47)). Unfortunately, the determination of these coefficients is generally done simultaneously with the unknown coefficients in the eigenfunction expansions for the decaying solution components by the stress free conditions

$$\sigma_{rr}^{kc}(a,\theta,z) = \sigma_{r\theta}^{kc}(a,\theta,z) = \sigma_{rz}^{kc}(a,\theta,z) = 0 \tag{60}$$

$$\sigma_{rr}^{k'c}(a,\theta,z) = \sigma_{r\theta}^{k'c}(a,\theta,z) = \sigma_{rz}^{k'c}(a,\theta,z) = 0 \tag{61}$$

Given the completeness of the eigenfunctions associated with the decaying state components, we may assign certain values to a_k^s and b_k^s for instance and then choose the Fourier coefficients in the decaying solution components to satisfy (60). However, the assigned values must be such that they do not violate the relative magnitude of the various stress components (found in various asymptotic analyses such as those referenced earlier). For example, it might appear reasonable to take a_k^s and b_k^s to satisfy $\sigma_{rr}^{kc}(a,\theta,z) = \sigma_{r\theta}^{kc}(a,\theta,z) = 0$ to leading order so that

$$\frac{\sigma_{rr}^{kc}(a,\theta,z)}{\cos(k\theta)} \sim -\frac{Ez}{1+v^2}\{b_k^s\beta_k a^k + a_k^s k(k-1)a^{k-2} + k(k+1)a^{-k-2}\} = 0 \tag{62}$$

$$\frac{\sigma_{r\theta}^{kc}(a,\theta,z)}{\sin(k\theta)} \sim \frac{Ez}{1+v}\{b_k^s k(k+1)a^k + a_k^s k(k-1)a^{k-2} - k(k+1)a^{-k-2}\} = 0 \tag{63}$$

where $\beta_k = (k+1)[(k+2) - v(k-2)]/(1-v)$, leaving the decaying components to absorb the residuals in $\sigma_{rr}^{kc}(a,\theta,z)$ and $\sigma_{r\theta}^{kc}(a,\theta,z)$ and to ensure $\sigma_{rz}^{kc}(a,\theta,z) = 0$. But this would make $\hat{\sigma}_{rr}^{kc}$ and $\hat{\sigma}_{r\theta}^{kc}$ of order h^3/a^3 while $\hat{\sigma}_{rz}^{kc}$ remains $O(h^2/a^2)$. This relative order of magnitude between the decaying components of both the transverse shear stress and the in-plane stresses is not consistent with the requirement of the differential equations of equilibrium and is therefore unacceptable. An acceptable assignment would be to require the two Kirchhoff contracted resultants to vanish (with the decaying components absorbing the residuals). The two conditions $M_{rr}^{kc}(a,\theta) = V_r^{kc}(a,\theta) = 0$ require:

$$\bar{a}_k^s k(k-1)a^{k-2} + \bar{b}_k^s \frac{k+1}{1-v}[(k+2) - v(k-2)]a^k + k(k+1)a^{-k} = 0 \tag{64}$$

$$\bar{a}_k^s k(k-1)a^{k-2} - \bar{b}_k^s (k+1)[4 - k(1-v)]a^k + k(k+1)a^{-k} = 0 \tag{65}$$

giving

$$a_k^s \sim \bar{a}_k^s = \frac{1-v}{3+v}(k+1)a^{-2k}, \quad b_k^s \sim \bar{b}_k^s = -\frac{1-v}{3+v}ka^{-2k-2} \tag{66}$$

and therewith

$$\hat{w}^{kc} = \frac{4}{3+v}a^{-k}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\hat{w}^{kc} = -\frac{4k}{3+v}a^{-k-1}. \tag{67}$$

Similarly, we have from $M_{rr}^{k'c}(a,\theta) = V_r^{k'c}(a,\theta) = 0$

$$\bar{a}_{k'}^s k(k-1)a^{k-2} + \bar{b}_{k'}^s \frac{k+1}{1-v}[(k+2) - v(k-2)]a^k + k(k+1)a^{-k} = 0 \tag{68}$$

$$\bar{a}_{k'}^s k(k-1)a^{k-2} - \bar{b}_{k'}^s (k+1)[4 - k(1-v)]a^k - k(k+1)a^{-k} = 0 \tag{69}$$

requiring

$$a_{k'}^s \sim \bar{a}_{k'}^s = \frac{k^2(1-v)^2 + 8(1+v)}{k(3+v)(1-v)}a^{-2k+2}, \quad b_{k'}^s \sim \bar{b}_{k'}^s = -\frac{1-v}{3+v}(k-1)a^{-2k} \tag{70}$$

and therewith

$$\hat{w}^{k'c} = \frac{4[k(1-v) + 2(1+v)]}{k(3+v)(1-v)}a^{-k+2}, \quad \frac{\mathrm{d}\hat{w}^{k'c}}{\mathrm{d}r} = -\frac{4[k(1-v) - 4]}{(3+v)(1-v)}a^{-k+1} \tag{71}$$

Note that our choice of $\{a_k^s, b_k^s, a_{k'}^s, b_{k'}^s\}$ by way of (64)–(69) does not alter the fact that Σ_{rr}^S, $\Sigma_{r\theta}^S$ and Σ_{rz}^S (for S corresponding to both k and k') all vanish at the plate edge as required by the specification of a (2) state; this is ensured by an appropriate choice of the decaying solution components for these singular states. Hence there is no new error accrued for this step in our method.

It is then straightforward to verify that the four desired relations (56)–(59) are in fact satisfied identically, and the conventional Kirchhoff contracted stress boundary conditions in fact apply to the leading term Levy interior solution:

$$M_{rr}(a,\theta) = \overline{M}_{rr}^{k}\cos(k\theta), \quad V_r(a,\theta) = \overline{V}_r^{k}\cos(k\theta) \tag{72}$$

We see then that the Kirchhoff contracted boundary conditions are in fact consistent with a leading term outer solution of the plate bending problem (known more conventionally as the Kirchhoff thin plate theory), at least for a circular plate with edge stresses proportional to $\cos(k\theta)$ and $\sin(k\theta)$ for an integer k (with the special cases of $k=0$ and $k=1$ requiring special treatment described earlier). The corresponding results for a circular plate with general edge loads follow immediately after a Fourier decomposition. However, we learned in Section 5 that the conventionally accepted stress boundary conditions for (the Kirchhoff theory of) plate bending may not be adequate for higher order plate theories. In the development above leading to the contracted boundary conditions (72), terms of order h/a have been omitted repeatedly, starting with the replacement of the exact singular (2) states $\{U_z^{kc}, U_r^{kc}, U_\theta^{kc}\}$ by $\{w^{kc}, \partial w^{kc}/\partial r, r^{-1}\partial w^{kc}/\partial\theta\}$ in (42). Hence, higher order accuracy in h/a offered by higher order plate equations may well be lost if the same contracted stress boundary conditions should be used for these higher order equations.

8. Concluding remarks

While the validity of the Kirchhoff contracted stress boundary conditions for the leading term (thin) plate theory has been established only for circular plate, much of the analysis above carries over to plates with an arbitrary (but simple) smooth edge and plates with more than one edge. An asymptotic expression for b_k similar to (42) but for a general smooth edge can be obtained in a similar development:

$$\begin{aligned} -8\pi Dk(k+1)b_k &= \int\int_E [\bar{\sigma}_{nn}U_n^{kc} + \bar{\sigma}_{nt}U_t^{kc} + \bar{\sigma}_{nz}U_z^{kc}]\,\mathrm{d}S \\ &\sim \int_\Gamma \left[-\overline{M}_{nn}\frac{\partial w^{kc}}{\partial n} - \overline{M}_{nt}\frac{\partial w^{kc}}{\partial s}w^{kc} + \overline{Q}_n w^{kc}\right]\mathrm{d}s \end{aligned} \tag{73}$$

where $\partial/\partial n$ and $\partial/\partial s$ are differentiation in direction normal and tangent to the edge curve Γ, respectively. Upon integration by parts and assuming all stress and displacement components are single-valued, we get

$$-8\pi Dk(k+1)b_k \sim \oint_\Gamma \left[-\overline{M}_{nn}\frac{\partial w^{kc}}{\partial n} + \overline{V}_n w^{kc}\right]\mathrm{d}s \tag{74}$$

where

$$\{\overline{M}_{nn}, \overline{M}_{nt}\} = \int_{-h}^{h}\{\bar{\sigma}_{nn}(s,z), \bar{\sigma}_{nt}(s,z)\}z\,\mathrm{d}z, \quad \overline{Q}_n = \int_{-h}^{h}\bar{\sigma}_{nz}(s,z)\,\mathrm{d}z$$

with the effective transverse shear resultant $\overline{V}_n$ is given by $\overline{V}_n = \overline{Q}_n + (\partial/\partial s)\overline{M}_{nt}$. A corresponding expression for a_k is

$$8\pi Dk(k+1)a_k \sim \oint_\Gamma \left[-\overline{M}_{nn}\frac{\partial w^{k'c}}{\partial n} + \overline{V}_n w^{k'c}\right]\mathrm{d}s \tag{75}$$

Even without going beyond the expressions for a_k and b_k, it should be evident from (75) and (74) that the leading term Levy (or Kirchhoff thin plate theory) solution is effectively determined by the bending moment resultant $\overline{M}_{nn}$ and the effective transverse shear resultant $\overline{V}_n$ of the three prescribed edge stress components $\{\bar{\sigma}_{nn}, \bar{\sigma}_{nt}, \bar{\sigma}_{nz}\}$, only these two edge resultant measures and no others. At the same time, the Kirchhoff contracted stress boundary conditions are not likely to be adequate for higher order plate theories, while the conditions (19) and (20) are always applicable. The possible inadequacy of the Kirchhoff contracted conditions for higher order interior (or outer) solutions has been known since the asymptotic results of Friedrichs and Dressler (1961) and Gol'denveizer (1962). However, unlike these and similar subsequent asymptotic analyses, the solutions (19) and (20) and more generally the approach initiated in Gregory and Wan (1985a) make it possible to avoid solving boundary value problems associated with the boundary layer components of the exact solution.

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